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Nonlocality and the Critical Reynolds Numbers of the Minimum State Magnetohydrodynamic Turbulence

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Magnetohydrodynamic (MHD) systems can be strongly nonlinear (turbulent) when their kinetic and magnetic Reynolds numbers are high, as is the case in many astrophysical and space plasma flows. Unfortunately these high Reynolds numbers are typically much greater than those currently attainable in numerical simulations of MHD turbulence. A natural question to ask is how can researchers be sure that their simulations have reproduced all of the most influential physics of the flows and magnetic fields? In this Report, a metric is defined to indicate whether the necessary physics of interest has been captured. It is found that current computing resources will typically not be sufficient to achieve this minimum state metric.

Introduction.—Magnetohydrodynamic (MHD) turbulence [1, 2] has been widely employed as a physical model in simulations and modeling of space physics and astrophysics systems. As is well known, the number of degrees of freedom in turbulent flows can be estimated using non-dimensional parameters such as the Reynolds number (Re) and magnetic Reynolds number (Rm). These can be interpreted as ratios of the nonlinear terms to the dissipative terms in the governing MHD equations. In space physics and astrophysics systems, estimates for Re and Rm are often in excess of 10^5 , sometimes by many orders of magnitude. Direct numerical simulation of such high Reynolds number systems would require resolutions that are well beyond what can be achieved using current and foreseeable supercomputers. Thus, it is highly desirable to determine whether the computationally feasible simulations—with much lower Re and Rm—still capture the most important physics of the flows of interest, despite the inevitable loss of information associated with the lower Reynolds numbers. Here we employ the *minimum state* concept [3] along with recent results on the wavenumber locality of nonlinear interactions in MHD turbulence [4] to estimate the minimum Reynolds numbers needed for accurate simulation of the energy-containing range in incompressible MHD turbulence.

The equations of incompressible three-dimensional MHD are

$$\frac{\partial \mathbf{u}}{\partial t} + \nu \nabla^2 \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{b} - \nabla p, \quad (1)$$

$$\frac{\partial \mathbf{b}}{\partial t} + \eta \nabla^2 \mathbf{b} = -\mathbf{u} \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{u}, \quad (2)$$

along with the solenoidality constraints $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \mathbf{b} = 0$ [e.g., 1]. Here, \mathbf{u} is the fluid velocity and \mathbf{b} the magnetic field, expressed in Alfvén speed units. Equations (1)–(2) are written so that the nonlinear terms are isolated on the RHS, along with the pressure gradient. Note that the nonlinear terms all have the same structure, $\sim \alpha \cdot \nabla \beta$, where α and β can be either \mathbf{u} or \mathbf{b} .

We begin by defining the basic requirement of a minimum state, namely capturing the key physics of the flow of interest. From an applications perspective the most important group of scales is often the energy-containing range. The integrity of the evolution of modes in this range can be protected by demanding that the (direct) interactions between

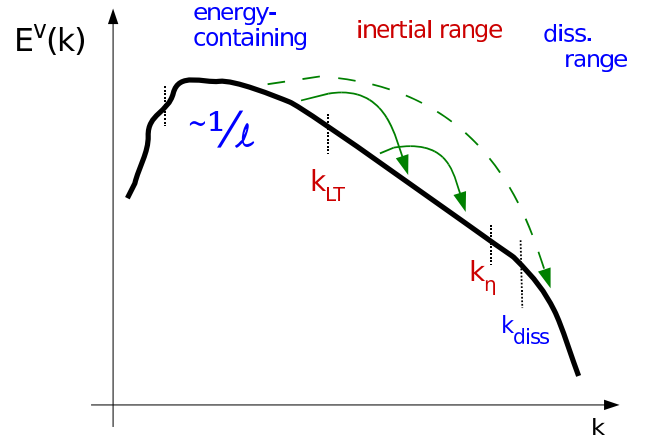


FIG. 1. (Color online) Sketch of a kinetic energy spectrum indicating the energy-containing, inertial, and dissipation ranges, and their wavenumber boundaries. The idea behind the minimum state is that the inertial range should be long enough so that direct interactions between modes in the energy-containing and dissipation ranges are energetically weak, indicated by the dashed (green) arrow. Some “strong” interactions are indicated via the solid (green) arrows.

them and modes in the dissipation range are weak [3]. In such situations the energy-containing and dissipation range scales will be separated by an inertial range, through which the energy originally resident at energy-containing scales cascades to smaller scales [5]. Moreover, the modes in the energy-containing range will then interact dominantly with themselves and modes in the inertial range. It seems likely that there will be critical values of Re and Rm below which this requirement cannot be satisfied. These Reynolds numbers define a *minimum state* flow. See Figure 1.

To quantify these ideas we will extend a criterion developed for Navier–Stokes (NS) turbulence [3] to the MHD case. Specifically, a minimum state flow is defined as one for which the (normalized) energy flux at the high- k end of the inertial range is half that at the low- k end [3], where $k = |\mathbf{k}|$ is the Fourier wavenumber. The remainder of the paper provides the necessary definitions and details required to obtain

the Reynolds numbers for a minimum state.

Lengthscales.—To calculate the minimum state we require estimates of the wavenumbers which bound the kinetic and magnetic inertial ranges. In particular, their scaling with Reynolds number is needed. Let ℓ denote the outer scale, or correlation length, of the velocity field, and let ℓ_B be the equivalent quantity for \mathbf{b} . Further, let \tilde{u} and \tilde{b} be characteristic rms values for the velocity and magnetic fields. Standard definitions of the (outer scale) kinetic and magnetic Reynolds numbers are $\text{Re} = \tilde{u}\ell/\nu$ and $\text{Rm} = \tilde{b}\ell/\eta$. We will also make use of an alternative magnetic Reynolds number, $\text{Rm}^* = \tilde{B}\ell_B/\eta$, which is based entirely on typical magnetic quantities [6]. The Kolmogorov dissipation scale for the kinetic energy is defined in the usual way as $\ell_{\text{diss}} = (\nu^3/\epsilon)^{1/4}$, where ϵ is the kinetic energy dissipation rate.

For NS turbulence, Dimotakis [7] suggested defining the inertial range as the set of scales which lie below the Liepmann–Taylor scale [8],

$$\lambda_{\text{LT}} = \frac{2\pi}{k_{\text{LT}}} \approx 5 \text{Re}^{-1/2} \ell, \quad (3)$$

and above the inner viscous scale [e.g., 9],

$$\lambda_\nu = \frac{2\pi}{k_\nu} \approx 50 \ell_{\text{diss}} \approx 50 \text{Re}^{-3/4} \ell. \quad (4)$$

Operationally, the latter is defined as the scale where the spectrum departs from the $\approx -5/3$ powerlaw [9–11]. The energy-containing range is thus treated as having $k \lesssim k_{\text{LT}}$, and the inertial range as $k \in [k_{\text{LT}}, k_\nu]$. We assume that the same boundaries hold for MHD turbulence (but see the final section) and define the magnetic versions analogously:

$$\zeta_{\text{LT}} = \frac{2\pi}{k_{\text{LT}}^B} \approx 5 \text{Rm}^{*-1/2} \ell, \quad (5)$$

$$\zeta_\eta = \frac{2\pi}{k_\eta} \approx 50 \text{Rm}^{*-3/4} \ell. \quad (6)$$

Energy Fluxes and Locality.—For turbulent systems, the flux of energy in Fourier space is a central concept, and numerous investigations of it have been performed for both NS [e.g., 12–14] and MHD [e.g., 4, 15–19] systems. Each of the nonlinear terms in Eqs. (1)–(2) is associated with such a flux, which we denote herein as $\Pi_{\alpha\beta}$. An important feature of the flux functions is their scaling with wavenumber, which provides information on the extent to which the contributing interactions are local in spectral space. The different scaling properties of these fluxes will be important in determining the minimum state Reynolds numbers.

Using direct numerical simulation databases, Domaradzki et al. [4] calculated normalized versions of the four energy flux functions, which they denoted as $\Pi_{\alpha\beta}(k|k_c)$. These represent the flux of energy to wavevectors with magnitudes greater than k_c , due to wavevector triads which have at least one member with a magnitude less than k (and normalized by the total flux through k_c for the particular $\alpha \cdot \nabla \beta$ term). Plotting these as a function of k/k_c reveals approximate powerlaw scaling for three of the four normalized fluxes. (See Figure 2

in [4].) Here we express their results in terms of the *scale disparity parameter* [13, 14],

$$s = \frac{\max(k, p, q)}{\min(k, p, q)}, \quad (7)$$

where k , p , and q are the magnitudes of wavevectors making up an interacting triad $\mathbf{k} = \mathbf{p} + \mathbf{q}$. This re-expression is convenient since $k/k_c \approx 1/s$ and thus, $\Pi_{\alpha\beta}(k|k_c) \approx \Pi_{\alpha\beta}(1/s)$. The scale disparity parameter is a measure of the elongation of the triads and has been used to characterize the degree of locality of interactions [e.g., 20].

The scalings observed by Domaradzki et al. [4] are

$$\Pi_{uu}(s) \sim \Pi_{ub}(s) \propto s^{-2/3}, \quad (8)$$

and

$$\Pi_{bb}(s) \propto s^{-1/3}. \quad (9)$$

The flux Π_{bb} is associated with removing the kinetic energy from the velocity field. It is the least local of these three flux functions. These numerical results are consistent with theoretical predictions [19]. Note that for NS turbulence, theory and simulations [13, 14, 21–27] suggest that $\Pi(s) \sim s^{-4/3}$, a scaling which is considerably more local than the MHD results.

The remaining flux function, Π_{bu} , is associated with the process of energy transfer to the magnetic field. The same study [4] found that it was non-universal and that it did not follow an s^{-M} scaling law. It does, however, decrease faster than Π_{uu} and Π_{ub} . While the reason for the different behavior of Π_{bu} is at present not clear, it is fortunate that the falloff is so steep since this suggests extremely local interactions for the term. Thus its detailed form will not affect the analysis here.

Minimum State.—For large enough Reynolds numbers, the energy-containing range of a turbulent flow will have very weak *direct* interactions with the dissipation range. As noted above, the minimum state is the lowest Reynolds number flow of this kind [3]. In flows that have a shorter inertial range, there will be significant direct interactions between the energy-containing and dissipation ranges, and the integrity of the energy-containing range modes will not be maintained. To ensure that strong direct couplings between these two ranges are absent we need to quantify what ‘strong’ means in this context, and then determine the length of the inertial range in a minimum state.

Here we define the direct interactions between the energy-containing and dissipation ranges as weak if the energy flux at the high- k end of the inertial range (e.g., at k_ν) is at most half that at the low- k end (e.g., at k_{LT}).

From numerical simulations, the peak of a normalized flux function, $\Pi(s_p)$, can be found, along with the value of s at which it occurs, s_p . Let s_h be the scale disparity parameter where the normalized flux reduces to half of its peak value, i.e., $\Pi(s_h) = \frac{1}{2} \Pi(s_p)$. The s^{-M} scaling properties of Eqs (8)–(9) lead to

$$\frac{s_h}{s_p} = 2^{(1/M)}. \quad (10)$$

This ratio can be used to determine the values of k_ν and k_η associated with the minimum state. The underlying idea is that Eq. (10) gives the length of the (minimum state) inertial range, in units of the Liepmann–Taylor wavenumber. Hence, we define $k_h = (s_h/s_p)k_{LT}$ and equate it to the high wavenumber end of the inertial range. For the momentum equation, the least local nonlinear term is Π_{bb} , yielding $s_h/s_p = 8$ and an inertial range wavenumber interval of $[k_{LT}, 8k_{LT}]$. For the induction equation we obtain $s_h/s_p = \sqrt{8} \approx 3$ and an inertial range of $[k_{LT}^B, 3k_{LT}^B]$.

We are now in position to calculate the critical Reynolds numbers for a minimum state. For the momentum equation we use Eqs. (3) and (4) in $8k_{LT} = k_\nu$, obtaining

$$\text{Re}_{\text{MS}} \approx 4.1 \times 10^7. \quad (11)$$

Proceeding similarly for the induction equation, $3k_{LT}^B = k_\eta$ yields

$$\text{Rm}_{\text{MS}}^* \approx 8.1 \times 10^5. \quad (12)$$

This is some 50 times smaller than Re_{MS} as a consequence of the more local nature of the nonlinear interactions in the induction equation. We recall that NS turbulence is even more local with $\Pi(s) \sim s^{-4/3}$ [3]. Using $2^{3/4} \approx 2$ in Eq. (10), this leads to a minimum state Reynolds number of 1.6×10^5 , which is significantly smaller than the MHD values [3].

Summary and Discussion.—We have extended the concept of a minimum state flow to the case of MHD turbulence, which is a widely used model in space physics and astrophysics applications. By insisting on the integrity of the energy-containing range dynamics, we have determined minimum Reynolds numbers for MHD simulations and experiments below which this condition is unlikely to be satisfied. These ‘critical’ values of Re and Rm^* are found to be rather large, as a consequence of the degree of nonlocality associated with the nonlinear terms in the MHD equations. As far as simulations of the minimum state are concerned, they are probably not feasible with current computing resources. However, they may become feasible within a few years.

Note that the numerical accuracy of a simulation for given Reynolds numbers is a distinct issue, relative to the above

discussed ‘physical integrity’ of a simulation. A recent exploration of the accuracy requirements for 2D MHD turbulence [28] concluded that sufficient accuracy is obtained if simulations retain wavenumbers a factor of three greater than the (Kolmogorov) dissipation wavenumber. If a smaller wavenumber range is retained then the accuracy of fourth-order (and higher-order) quantities like the kurtosis was compromised.

In closing we briefly mention some possible extensions and complications associated with the isotropic MHD model employed above. As is well known, the presence of an energetic large-scale (e.g., mean) magnetic field (\mathbf{B}_0) induces anisotropy in \mathbf{u} and \mathbf{b} [29–32]. This anisotropy could result in somewhat different critical Reynolds numbers, although the qualitative results presented herein would likely still hold.

Another issue, is our assumption that the NS inertial range boundaries carry over essentially unchanged to the MHD case. However, in MHD the inertial range boundaries may scale a little differently than they do in the NS case. For example, the energy spectrum has a bottleneck feature near the dissipation scale [e.g., 33], but this appears to be more pronounced in NS turbulence than in MHD turbulence [e.g., 34]. Thus, our estimates for k_ν and k_η could be argued to be too small, leading to estimates for the critical Reynolds numbers which are too large.

Finally, we emphasize that in actual space physics and astrophysics systems the nature of the dissipation mechanisms may be quite different from the uniform viscous and resistive dissipation of Eqs. (1)–(2). In particular, the dissipation scales are not expected to be universal. Plasma effects, such as damping by waves at ion and/or electron gyroradii or inertial lengths may be important [35–38].

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